Circular Waveguide Taper of Improved Design

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Transitions between round waveguides of different sizes for TE_{01} transmission are required to be free of mode conversion. Conical tapers with a gradual change of cone angle transform cylindrical waves in the round waveguide into spherical waves in the transition region. They thus cause very little power conversion to spurious modes. Optimal tapers and almost optimal tapers are found. An improved taper connecting $\frac{7}{8}$ in. to 2 in. I.D. *waveguide must be 3 ft long to keep the spurious mode level below −50 db for frequencies up to 75 kmc. A taper of constant cone angle would have to be 58 ft long to obtain the same spurious mode level.*

I. INTRODUCTION

In long-distance waveguide transmission, multimode waveguides of large diameter must be used to exploit the low-loss properties of the circular electric wave. Multiplexing of a series of frequency bands into one pipe is, however, most easily done at certain smaller diameters. Likewise, sharp intentional bends can be negotiated more easily at small diameters. Therefore, transitions between different diameters must be made quite frequently in a waveguide system.

If these transitions are built in the form of a conical taper which matches the waveguide sizes of both sides, they tend to excite higher circular electric waves. Since no simple means are known to suppress higher circular electric waves without affecting the lowest circular electric wave, mode conversion-reconversion distortion can only be avoided by keeping power in all the higher circular electric waves at an extremely low level. Transitions are therefore required which will introduce very little conversion to higher circular electric modes. Conical transitions with a constant cone angle, unless they are made very long, generally excite too high a level of these spurious modes.

Wave propagation in the conical taper is most easily explained in terms of normal modes. The normal modes of the circular electric wave family — the only family of interest here — are, in the cylindrical guide, cylindrical circular waves with plane equiphase surfaces and, in the conical guide, spherical circular waves with spherical equiphase surfaces. At the junction of a cylindrical guide to a conical guide — such as occurs twice in a conical transition — a cylindrical wave from the cylindrical guide excites a series of spherical waves in the conical guide. For instance, an incident TE_{01} wave will excite all the TE_{0m} waves, thus causing a rather high spurious mode level.

To avoid this mode conversion, a transition which transforms the cylindrical wave into the spherical wave must be introduced at the junction. S. P. Morgan[1] has suggested and worked out the design of dielectric inserts placed near the junction which, acting as quasi-optical lenses, transform the cylindrical waves into the spherical waves. However, because of the dispersive character of the lenses and of the waveguide, good broadband performance is difficult to achieve.

Another way of making a transition from cylindrical waves to spherical waves — at least approximately — is to taper the cone angle from zero at the cylindrical guide to the finite value of the conical guide. If this is done gradually enough, nearly all the power incident in the cylindrical wave will be transformed into the spherical wave, with a very low spurious mode level. Although this is only an approximate solution to the problem, it does have good broadband characteristics.

II. WAVE PROPAGATION IN A TAPERED WAVEGUIDE

In a tapered circular waveguide such as the one shown in Fig. 1, the radius a is related to the distance z along the taper by a function which is assumed only to be smooth and to have a zero first derivative at both ends.

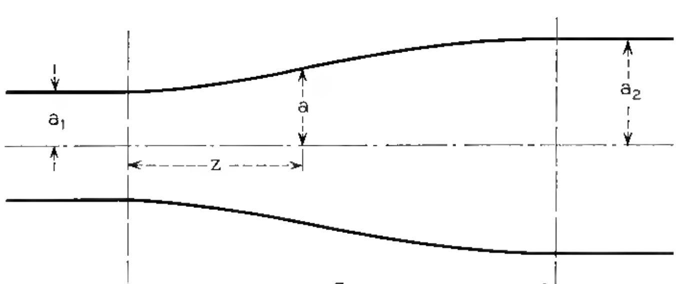

Fig. 1 — Tapered waveguide transition.

The field excited in any cross section by an incident TE_{01} wave can be expressed as the sum of the TE_{0m} waves of a cylindrical guide having the same radius as that of the cross section. With this representation, the tapered waveguide appears to be an infinite set of mutually coupled transmission lines, each line representing one of the cylindrical TE_{0m} waves. The wave propagation is described by an infinite set of first-order differential equations. If we assume the taper to be gradual enough, the power in all TE_{0m} terms with m greater than 1 will be small compared to the power in the TE_{01} term. We may then consider only coupling between TE_{01} and one of the TE_{0m} terms at a time. Furthermore, we need consider only the forward travelling waves, since the relative power coupled from the forward waves to the backward waves is quite small. Thus, the infinite system is reduced to the well-known coupled line equations:[2]

$$\frac{dA_1}{dz} = -j\beta_1 A_1 + k_{21} A_2 , \quad \frac{dA_2}{dz} = -j\beta_2 A_2 + k_{12} A_1 , \tag{1}$$

in which $A_{1,2}$ are the complex amplitudes of the cylindrical TE_{01} and TE_{0m} respectively, $\beta_{1,2}$ are the phase constants of these cylindrical modes and k_{21}, k_{12} are the coupling coefficients. The coupling coefficients are calculated in the Appendix by converting Maxwell's equations into generalized telegraphist's equations. Both the phase constants and the coupling coefficients are functions of the distance along the taper. Power conservation requires

$$\begin{gathered} | k_{12} | = k = | k_{21} | \\ k_{12} k_{21} = -k^2. \end{gathered} \tag{2}$$

Coupled line equations with varying coupling coefficient and varying phase constant have been treated by Louisell.[3]

In order to solve (1), local normal modes must be introduced. Normal modes are waveforms in a uniform waveguide which propagate without change of field configuration or, in terms of the coupled line description, do not couple mutually. Analogously, local normal modes in a nonuniform waveguide are waveforms of a local cross section which would propagate without change of field configuration in a waveguide which is uniform with respect to that local cross section. For example, local normal modes in the waveguide taper are the spherical waves of a conical waveguide with a cone angle corresponding to the local cone angle in the taper. They would propagate in the conical waveguide without mutual coupling.

For the system of equations (1) local normal modes are defined by

$$
\begin{aligned}
A_1 &= \left(W_1 \cos\frac{1}{2}\xi - W_2 \sin\frac{1}{2}\xi\right) \exp\left(-j\int_0^z \beta\, dz\right), \\
A_2 &= -\sqrt{\frac{k_{12}}{k_{21}}}\left(W_1 \sin\frac{1}{2}\xi + W_2 \cos\frac{1}{2}\xi\right) \exp\left(-j\int_0^z \beta\, dz\right),
\end{aligned} \tag{3}
$$

in which

$$\beta = \frac{1}{2}(\beta_1 + \beta_2), \qquad \tan\xi = 2\frac{k}{\beta_1 - \beta_2} = 2\frac{k}{\Delta\beta}. \tag{4}$$

Upon substituting (3) into equations (1), the local normal modes must satisfy

$$\frac{dW_1}{dz} + j\Gamma W_1 = \frac{1}{2}\frac{d\xi}{dz} W_2, \qquad \frac{dW_2}{dz} - j\Gamma W_2 = -\frac{1}{2}\frac{d\xi}{dz} W_1, \tag{5}$$

where

$$\Gamma(z) = \tfrac{1}{2}\sqrt{\Delta\beta^2 + 4k^2}.$$

Equations (5) are coupled only through the terms proportional to $d\xi/dz$. If ξ is constant, they reduce to uncoupled equations for W_1 and W_2. For a gentle change in taper angle, with

$$\left|\frac{1}{2\Gamma}\frac{d\xi}{dz}\right| \ll 1, \tag{6}$$

approximate solutions of (5) which proceed essentially in powers of $d\xi/dz$ can be written down:

$$
\begin{aligned}
W_1(z) &= e^{-j\rho(z)}\Bigg[W_1(0) + \frac{1}{2}W_2(0)\int_0^z \frac{d\xi}{dz'} e^{2j\rho(z')}\, dz' \\
&\qquad - \frac{1}{4}W_1(0)\int_0^z \frac{d\xi}{dz'} e^{2j\rho(z')}\int_0^{z'} \frac{d\xi}{dz''} e^{-2j\rho(z'')}\, dz''\, dz'\Bigg], \\
W_2(z) &= e^{j\rho(z)}\Bigg[W_2(0) - \frac{1}{2}W_1(0)\int_0^z \frac{d\xi}{dz'} e^{-2j\rho(z')}\, dz' \\
&\qquad - \frac{1}{4}W_2(0)\int_0^z \frac{d\xi}{dz'} e^{-2j\rho(z')}\int_0^{z'} \frac{d\xi}{dz''} e^{2j\rho(z'')}\, dz''\, dz'\Bigg],
\end{aligned} \tag{7}
$$

in which

$$\rho(z) = \int_0^z \Gamma(z')\, dz'. \tag{8}$$

The initial conditions in the taper are $A_1(0) = 1$ and $A_2(0) = 0$. The taper begins with zero cone angle; hence, from (3),

$$W_1(0) = 1 \qquad W_2(0) = 0,$$

in which W_1 corresponds to the TE_{01} wave and is the wanted local normal mode, while W_2 corresponds to one of the TE_{0n} waves with $n \neq 1$ and is an unwanted mode. At the end z_1 of the taper, the unwanted mode amplitude is

$$|W_2(z_1)| = \frac{1}{2}\left| \int_0^{z_1} \frac{d\xi}{dz} e^{-2j\rho(z)}\, dz \right|. \tag{9}$$

At the taper end the cone angle is again zero, $(\xi(z_1) = 0)$. Therefore $|W_2(z_1)|$ equals $|A_2(z_1)|$. Equation (9) integrated by parts becomes

$$|W_2(z_1)| = \left| \int_0^{z_1} \Gamma \xi e^{-2j\rho(z)}\, dz \right|. \tag{10}$$

The mode conversion in a smooth but otherwise arbitrary taper can be calculated with (9) or (10).

III. DESIGN OF A TAPER

A waveguide taper can always be built to have as low a mode conversion as is wanted in a certain frequency band merely by making it long enough. However, an optimally designed taper has the smallest possible length for a given difference in diameters at its two ends and for a specified unwanted mode level in a given frequency band.

Some analogies between this problem and the problem of a transmission line taper of optimum design are evident. The transmission line taper for matching impedances is nothing but a tapered waveguide in which only one mode is propagating. Power can only be converted into reflected waves, and it is this reflected power which is kept small in a properly designed transmission line taper. If more than one mode is propagating, power will be scattered not only into the reflected wave but also into the other propagating modes. In fact, the power scattered into backward traveling waves is quite small compared to the power scattered into forward traveling waves, and only the latter need be considered in a multimode waveguide taper. Therefore, the mode conversion in the waveguide transition corresponds to the reflection in the transmission line taper.

It has been shown that a transition between transmission lines of different characteristic impedances is optimally made by a series of steps spaced about a quarter wavelength apart.[4] The magnitude of these steps is chosen to give an input reflection described in its frequency dependence by a Tschebycheff polynomial. Similarly, a conical waveguide transition is expected to perform optimally when it is composed of a number of sections with different cone angles, as in Fig. 2. The lengths of these sections should be chosen so that the converted energy from adjacent joints of sections is 180 degrees out of phase, and the changes of angles from section to section should be chosen so that the over-all conversion pattern corresponds to a Tschebycheff polynomial. However, since there is more than one mode to which power is converted and since the phase constants change along the transition, the design of such a multisection transition will be very difficult, if not impossible.

If the number of sections in the line transition is allowed to increase indefinitely for a fixed over-all length, a continuous transmission line taper is formed. The results of the multistep transition of optimum design have been extended to this case.[5] The input reflection of the continuous taper of optimum design is described in its frequency dependence by a Tschebycheff polynomial of infinite degree. The taper curve itself is given essentially by the Fourier transform of this Tschebycheff polynomial. The frequency band of constant low reflection now extends from a certain lowest frequency to infinity.

The transmission line taper of optimum design does not have a continuous reflection distribution, but it has reflection impulses at both ends. Assuming a completely continuous reflection distribution, an almost optimum design has been found.[6] The raised cosine function is a reflection distribution which, in practical cases, closely approximates in its frequency dependence the input reflection of the continuous taper of almost optimum design.

If the integration in (9) for the mode conversion in the waveguide

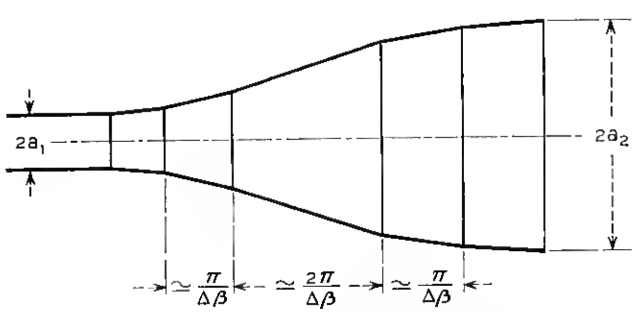

Fig. 2 — Multisection waveguide transition.

taper is extended over the parameter ρ,

$$|W_2(\rho_1)| = \left| \int_0^{\rho_1} \xi(\rho) e^{-2j\rho} \, d\rho \right|, \tag{11}$$

the relationship of the mode conversion in the multimode waveguide taper to the input reflection of the transmission line taper becomes evident. In (11) ρ corresponds to the longitudinal coordinate and $\xi(\rho)$ to the reflection distribution of the transmission line taper.

The reflection distribution for an almost optimally designed line taper can be substituted for $\xi(\rho)$ to give the almost optimally designed waveguide taper. Even the optimum design can be found when some care is exercised in using steps in the waveguide taper to simulate the impulse functions in the reflection distribution.

Once $\xi(\rho)$ is given, we can find the waveguide radius $a(\rho)$ and the axial distance $z(\rho)$ from (4), (5) and (8). The coupling coefficient can be written

$$k = c(a) \frac{da}{dz} \equiv \Gamma \, c(a) \frac{da}{d\rho} . \tag{12}$$

Therefore,

$$c(a) \frac{da}{d\rho} = \frac{\tan \xi(\rho)}{\sqrt{1 + \tan^2 \xi(\rho)}}, \tag{13}$$

which can readily be integrated by separation of variables to give $a(\rho)$. With a known, Γ is also known as a function of ρ, and the axial distance can be calculated:

$$z = \int_0^{\rho} \frac{d\rho}{\Gamma(\rho)} . \tag{14}$$

Some restricting remarks must be made in concluding this outline of the taper design.

Only one of the coupled higher-order modes has been considered here. This is adequate as long as the cone angle changes only gradually. Since all other higher order modes are weaker in coupling and farther removed in phase constant, the conversion to these modes will always be much smaller than the conversion to the TE_{02} wave. If, however, there are abrupt changes in cone angle or even if the diameter of the guide changes in steps, as in the waveguide taper of optimum design, the power conversion to higher modes must be checked.

The design procedure outlined here works only as long as the coupled mode does not go through cutoff within the transition. Power converted to backward travelling waves has been neglected, since the coupling to these waves is smaller and they are much more removed in phase constant than are the forward travelling waves. In the vicinity of cutoff, this is no longer true. Here, the propagation is more suitably described by the original generalized telegraphist's equations (35) and (36) of the Appendix. They are, however, not easily solved in the cutoff region of a mode.

Since the coupling coefficients in (35) and (36) do not change with frequency, and since the difference in propagation constants of the coupled waves does not increase with decreasing frequency, the power conversion does not increase when the frequency is lowered to bring a coupled mode to cutoff within the taper. Consequently, the waveguide taper can be designed for a frequency high enough to keep the lowest coupled mode propagating throughout the taper. It will then work properly at all lower frequencies.

IV. THE RAISED COSINE TAPER

As a characteristic example, we will design a taper whose conversion distribution follows the raised cosine function

$$\xi(\rho) = \frac{2C}{\rho_1} \sin^2 \pi \frac{\rho}{\rho_1}, \tag{15}$$

in which C is yet to be determined from the radii a_1 and a_2. According to (11) the over-all conversion will then be given by the Fourier transform of the raised cosine function

$$\frac{|W_2(\rho_1)|}{C} = \frac{|1 - e^{-2j\rho_1}|}{2\rho_1\left(1 - \dfrac{\rho_1^2}{\pi^2}\right)}. \tag{16}$$

Equation (16) is plotted in Fig. 3.

We assume the TE_{01} and TE_{02} modes to be far enough from cutoff so that

$$\left(\frac{\lambda}{\lambda_c}\right)^2 \ll 1 \tag{17}$$

for both modes throughout the taper. Furthermore, we assume the taper to be gentle enough so that

$$\xi^2 \ll 1. \tag{18}$$

These assumptions will enable us to carry out the integrations analytically.

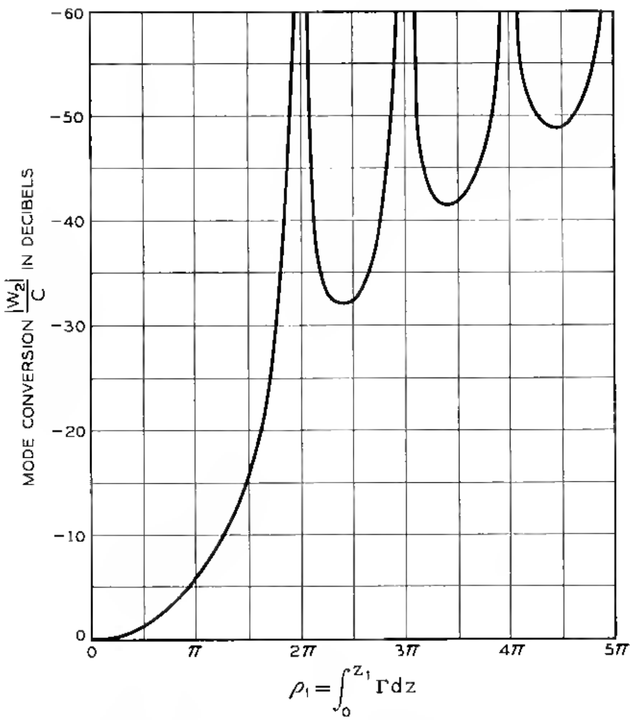

Fig. 3 — Mode conversion in the raised cosine taper.

Integrating (13) with

$$c = \frac{2k_1k_2}{k_2^2 - k_1^2}\frac{1}{a}$$

[from (42) and (12)] and $\xi(\rho)$ from (15) we get

$$\ln\frac{a}{a_1} = \ln\frac{a_2}{a_1}\left(\frac{\rho}{\rho_1} - \frac{1}{2\pi}\sin 2\pi\frac{\rho}{\rho_1}\right), \tag{19}$$

where C has been eliminated by the boundary conditions $a(0) = a_1$ and $a(\rho_1) = a_2$:

$$C = \frac{2k_1k_2}{k_2^2 - k_1^2}\ln\frac{a_2}{a_2}. \tag{20}$$

Because of (17) and (18)

$$\Gamma = \frac{k_2^2 - k_1^2}{2\beta_0 a^2}. \tag{21}$$

The axial distance is found from (14) by substituting (19) for a in (21):

$$z = \frac{4\beta_0 a_1^2}{k_2^2 - k_1^2} \int_0^\rho \exp\left[2 \ln \frac{a_2}{a_1}\left(\frac{\rho}{\rho_1} - \frac{1}{2\pi} \sin \frac{2}{\pi} \frac{\rho}{\rho_1}\right)\right] d\rho, \qquad (22)$$

where β_0 is the phase constant of free space. With $\alpha = 1/\pi \ln a_2/a_1$ and $x = 2\pi(\rho/\rho_1)$, the integrand can be expanded in a series

$$z = \frac{2\beta_0 a_1^2 \rho_1}{\pi(k_2^2 - k_1^2)} \int_0^x e^{\alpha x}\left(1 - \alpha \sin x + \frac{\alpha^2}{2} \sin^2 x \mp \cdots\right) dx \quad (23)$$

and, for moderate values of α, the axial distance z can be calculated by term-by-term integration. The total length of the taper is approximately

$$z_1 \cong \rho_1 \frac{2\beta_0}{\ln \dfrac{a_2}{a_1}} \frac{a_2^2 - a_1^2}{k_2^2 - k_1^2}\left[\frac{\pi^2 + 2 \ln^2 \dfrac{a_2}{a_1}}{\pi^2 + \ln^2 \dfrac{a_2}{a_1}} + \frac{\ln^2 \dfrac{a_2}{a_1}}{4\pi^2 + \ln^2 \dfrac{a_2}{a_1}}\right]. \qquad (24)$$

A numerical example will show the advantage of a properly tapered waveguide transition. Suppose a transition designed for the TE_{01} wave and connecting $\frac{7}{8}$ in. I.D. pipe to 2 in. I.D. pipe is required to have a spurious mode level of less than -50 db for all frequencies up to 75 kmc. The most seriously coupled wave is TE_{02}. For a raised cosine taper we get, from (20), C equal to 1.28 and consequently, from Fig. 3, ρ_1 equal to 14.8. Hence, from (24), the required taper length is z_1 equal to 3 ft. The taper curve is plotted in Fig. 4. This curve has been calculated from (19) and (23), using ρ as a parameter. A taper of constant cone angle satisfying the same requirements would have to be 58 ft. long.[1]

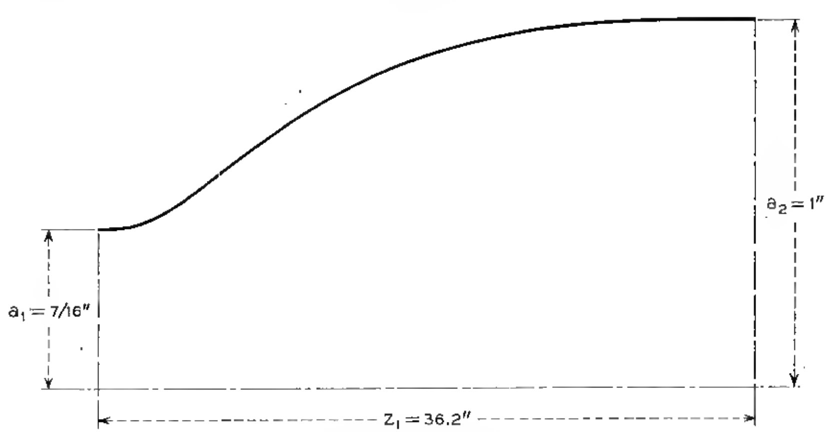

Fig. 4 — Raised cosine taper.

Circular waveguide tapers have been built according to Fig. 4. Their TE_{01}-TE_{02} conversion loss was measured at frequencies near 55 kmc and found to be higher than 50 db.

V. SUMMARY

The power conversion from TE_{01} to higher circular electric waves occurring in conical transitions of round waveguide can be reduced by changing the cone angle gradually rather than abruptly. Instead of a taper with constant cone angle, a transition with changing cone angle, which transforms the cylindrical waves of the round waveguide into spherical waves in the transition region, is suggested.

Upon converting Maxwell's equations into generalized telegraphist's equations, the transition is represented as a set of nonuniform transmission lines, nonuniformly coupled. With proper choice of the coupling distribution — and hence the cone angle — as a function of the distance along the transition, we can find an optimum design which minimizes the length of the transition for a specified frequency range and spurious mode level. In a transition of optimum design the mode conversion is given by a Tschebycheff polynomial of infinite degree in its frequency dependence, and the geometry of the transition is found from the Fourier transform of this Tschebycheff polynomial. A simpler design, but still a good one, has a transition geometry given essentially by a raised cosine function.

APPENDIX

Generalized Telegraphist's Equations of the Circular Electric Waveguide Taper

A very convenient mathematical formulation of the electromagnetic problem in the waveguide transition is provided by S. A. Schelkunoff's generalized telegraphist's equations for waveguides.[7] Maxwell's equations for transverse electric and circular symmetric waves are, in cylindrical coordinates:

$$\frac{\partial E_\varphi}{\partial z} = j\omega\mu H_r, \tag{25}$$

$$\frac{\partial H_r}{\partial z} = j\omega\epsilon E_\varphi + \frac{\partial H_z}{\partial r}, \tag{26}$$

$$\frac{\partial}{\partial r}(rE_\varphi) = -j\omega\mu r H_z, \tag{27}$$

where E_φ, H_z and H_r are the only nonvanishing components of the field, ϵ is the dielectric permittivity, μ the magnetic permeability and ω the angular pulsation. The exponential dependence of time is understood.

The boundary conditions of the waveguide taper are, at r equal to a:

$$E_\varphi = 0 \tag{28}$$

$$H_r = \frac{da}{dz} H_z . \tag{29}$$

The field at any cross section of the paper is represented as a superposition of the fields of the normal modes in a cylindrical guide of the same cross section:

$$E_\varphi = \sum_m V_m \frac{J_1\left(k_m \frac{r}{a}\right)}{\sqrt{\pi} a J_0(k_m)}, \tag{30}$$

$$H_r = -\sum_m I_m \frac{J_1\left(k_m \frac{r}{a}\right)}{\sqrt{\pi} a J_0(k_m)}, \tag{31}$$

$$H_z = \sum_m i_m \frac{J_0\left(k_m \frac{r}{a}\right)}{\sqrt{\pi} a J_0(k_m)}, \tag{32}$$

where J_0 and J_1 are Bessel functions of the first kind and k_m is the mth zero of J_1. The V_m, I_m and i_m have the dimensions of voltages and currents. The factors of V_m and I_m are normalized so that $P_m = 1/2(V_m I_m^*)$ is the complex power flow in each normal mode. It has to be kept in mind that a is a function of z.

The boundary condition (28) is satisfied by the individual terms of the series for E_φ. Hence, this series converges uniformly. Not so the series for H_r: (31) is a representation for H_r only in the open interval $0 \leqq r < a$, since the individual terms vanish at $r = a$ but, according to (29), H_r does not. In the closed interval, the series (31) represents a discontinuous function and therefore does not converge uniformly. Term-by-term differentiation will make the series diverge.

The relationship between i_m and V_m is found by substituting in (27) the series (30) for E_φ and the series (32) for H_z, and comparing coefficients

$$i_m = j \frac{k_m}{\omega \mu a} V_m . \tag{33}$$

Using this relation and substituting the series (30) for E_φ in (25) at $r = a$, we find that the boundary condition (29) is satisfied.

To convert Maxwell's equations into generalized telegraphist's equations, we introduce (30) and (31) into (25) and (26), multiply both sides of both equations by $J_1[k_n(r/a)]$ and integrate over the cross section. Since the series for H_r does not converge uniformly, we write for the left-hand side of (26)

$$rJ_1\left(k_n \frac{r}{a}\right)\frac{\partial H_r}{\partial z} = -rH_r \frac{\partial}{\partial z}\left[J_1\left(k_n \frac{r}{a}\right)\right] + \frac{\partial}{\partial z}\left[rJ_1\left(k_n \frac{r}{a}\right)H_r\right] \quad (34)$$

and invert integration and differentiation in the second term of this expression.

The generalized telegraphist's equations have the following form:

$$\frac{dV_n}{dz} = -j\omega\mu I_n + \frac{1}{a}\frac{da}{dz}\sum_m \frac{2k_nk_m}{k_m^2 - k_n^2} V_m , \quad (35)$$

$$\frac{dI_n}{dz} = -j\frac{\beta_n^2}{\omega\mu} V_n + \frac{1}{a}\frac{da}{dz}\sum_m \frac{2k_nk_m}{k_m^2 - k_n^2} I_m . \quad (36)$$

The summations are extended over all m except $m = n$. The quantity β_n is the phase constant of the nth mode in a cylindrical guide of the particular cross section; β_n is a function of a and therefore of z.

The generalized telegraphist's equations represent an infinite set of coupled nonuniform transmission lines. For our purpose, it is convenient to write the transmission-line equations not in terms of currents and voltages, but in terms of the amplitudes of forward and backward travelling waves. Thus, let A and B be the amplitudes of the forward and backward waves of a typical mode at a certain cross section. The mode current and voltages are related to the wave amplitudes by

$$V = \sqrt{K}\,(A + B), \quad (37)$$

$$I = \frac{1}{\sqrt{K}}(A - B), \quad (38)$$

where K is the wave impedance

$$K_n = \frac{\omega\mu}{\beta_n}. \quad (39)$$

If the currents and voltages in the generalized telegraphist's equations (35) and (36) are represented in terms of the traveling-wave amplitudes, after some obvious additions and subtractions, the following equations for coupled traveling waves are obtained:

$$\frac{dA_n}{dz} = -j\beta_n A_n - \frac{1}{2}\frac{K_n'}{K_n} B_n + \sum_m (k_{nm}^+ A_m + k_{nm}^- B_m) \tag{40}$$

$$\frac{dB_n}{dz} = j\beta_n B_n - \frac{1}{2}\frac{K_n'}{K_n} A_n + \sum_m (k_{nm}^+ B_m - k_{nm}^- A_m). \tag{41}$$

The k's are coupling coefficients defined by:

$$k_{nm}^{\pm} = \frac{k_n k_m}{k_m^2 - k_n^2}\left(\sqrt{\frac{K_n}{K_m}} \pm \sqrt{\frac{K_m}{K_n}}\right)\frac{1}{a}\frac{da}{dz} \tag{42}$$

and

$$K_n' = \frac{dK_n}{dz}.$$

For a cylindrical guide, da/dz is equal to zero and (40) and (41) reduce to uncoupled transmission-line equations.

REFERENCES

1. Morgan, S. P., Control of Spurious Modes in Multimode Waveguides by Use of Foam Dielectric Inserts, presented orally at the May 1956 Joint Meeting of U.R.S.I. and I.R.E.
2. Miller, S. E., Coupled Wave Theory and Waveguide Applications, B.S.T.J., **33,** May 1954, pp. 661–719.
3. Louisell, W. H., Analysis of the Single Tapered Mode Coupler, B.S.T.J., **34,** July 1955, pp. 853–870.
4. Burkhardtmaier, W., Widerstandstransformation mit Leitungen, Funk and Ton, March 1949, pp. 151–157, and April 1949, pp. 202–213; Collin, R. E., The Theory and Design of Wideband Multisection Quarter-Wave Transformers, Proc. I.R.E., **43,** February 1955, pp. 179–185.
5. Klopfenstein, R. W., A Transmission Line Taper of Improved Design, Proc. I.R.E., **44,** January 1956, pp. 31–35.
6. Collin, R. E., The Optimum Tapered Transmission Line Matching Section, Proc. I.R.E., **44,** April 1956, pp. 539–548.
7. Schelkunoff, S. A., Conversion of Maxwell's Equations into Generalized Telegraphist's Equations, B.S.T.J., **34,** September 1955, pp. 995–1043.